

# TOPOLOGICAL COMPLEXITY OF $n$ POINTS ON A TREE

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**ABSTRACT.** The topological complexity of a path-connected space  $X$ , denoted  $TC(X)$ , can be thought of as the minimum number of continuous rules needed to describe how to move from one point in  $X$  to another. The space  $X$  is often interpreted as a configuration space in some real-life context. Here, we consider the case where  $X$  is the space of configurations of  $n$  points on a tree  $\Gamma$ . We will be interested in two such configuration spaces. In the first, denoted  $C^n(\Gamma)$ , the points are distinguishable, while in the second,  $UC^n(\Gamma)$ , the points are indistinguishable. We determine  $TC(UC^n(\Gamma))$  for any tree  $\Gamma$  and many values of  $n$ , and consequently determine  $TC(C^n(\Gamma))$  for the same values of  $n$  (provided the configuration spaces are path-connected).

## 1. INTRODUCTION

For any topological space  $X$ , let  $P(X)$  be the space of continuous paths  $\sigma : [0, 1] \rightarrow X$  equipped with the compact-open topology. There is a fibration  $p : P(X) \rightarrow X \times X$  which sends a path  $\sigma$  to its endpoints:  $p(\sigma) = (\sigma(0), \sigma(1))$ . When studying the problem of motion planning within a topological space, one often wishes to find sections of this fibration. That is, one wishes to find functions  $s : X \times X \rightarrow P(X)$  such that  $p \circ s$  is the identity. Such a function takes a pair of points as input and produces a path between those points, hence the relation to motion planning. The continuity of a section  $s$  at a point  $(x, y) \in X \times X$  means that if  $(x', y')$  is “close” to  $(x, y)$ , then the path  $s(x', y')$  is “close” to the path  $s(x, y)$ . Unfortunately it is a rarity that such a function can be continuous over all of  $X \times X$  (in fact, such a continuous section exists if and only if the space  $X$  is contractible, see [3]). This leads to the definition of topological complexity introduced by Farber in [3]:

**Definition 1.1.** *For any path-connected space  $X$ , the topological complexity of  $X$ , denoted  $TC(X)$ , is the smallest integer  $k \geq 1$  such that there is a cover of  $X \times X$  by open sets  $U_1, U_2, \dots, U_k$  and continuous sections  $s_i : U_i \rightarrow P(X)$ . If there is no such  $k$ , set  $TC(X) = \infty$ .*

Such a collection of sets  $U_i$  and sections  $s_i$  is called a *motion planning algorithm*. Thus,  $TC(X)$  is in some sense the smallest number of continuous rules required to describe how to move between any two points in  $X$ . The space  $X$  is often viewed as the space of configurations of some real-world system. One example is when  $X$  is the space of configurations of  $n$  robots which move around a factory along a system of one-dimensional tracks. Such a system of tracks can be interpreted as a graph  $\Gamma$  (a one-dimensional CW complex). There are two types of these configuration spaces: in the first, denoted  $C^n(\Gamma)$ , the robots are distinguishable,

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and in the second, denoted  $UC^n(\Gamma)$ , the robots are indistinguishable. In other words, in  $C^n(\Gamma)$ , both the points of  $\Gamma$  occupied by robots, and the specific robots which occupy those points are of interest, while in  $UC^n(\Gamma)$ , it is only required that each specified point in  $\Gamma$  is occupied by *some* robot, but it is irrelevant which specific robot occupies each point. There are different real-world situations in which one configuration space is preferable over the other, depending on whether or not the robots are to perform different tasks. Our main goal is to study the topological complexity of the configuration spaces  $C^n(\Gamma)$  and  $UC^n(\Gamma)$  when  $\Gamma$  is a tree (a tree is a connected graph which has no cycles). The topological complexity of these configuration spaces is related to the number of vertices of degree greater than 2 (the degree of a vertex is defined in Section 2). These vertices are called *essential vertices*, and  $m(\Gamma)$  is the number of essential vertices in  $\Gamma$ . Here, an *arc* in  $\Gamma$  is a subspace homeomorphic to a non-trivial closed interval. We will be interested in certain collections of arcs in  $\Gamma$  which are called *allowable* and will be defined in Definition 2.3. In Section 3, we establish the following:

**Theorem 1.2.** *Let  $\Gamma$  be a tree with  $m := m(\Gamma) \geq 1$ .*

- (1) *Let  $k$  be the smallest integer such that there is a collection of oriented arcs  $\{A_i\}_{i=1}^k$  which is allowable for the collection of all vertices of degree 3 in  $\Gamma$ . If there are no vertices of degree 3, let  $k = 0$ . Let  $n \geq 2m + k$  be an integer. Then,  $TC(C^n(\Gamma)) = TC(UC^n(\Gamma)) = 2m + 1$ .*
- (2) *Let  $n = 2q + \epsilon < 2m$ , with  $\epsilon \in \{0, 1\}$  and  $q \geq 1$ , let  $r$  be the number of vertices of degree greater than 3, and let  $s$  be the number of vertices of degree 3. If*
  - (a)  *$s \geq 2(q - r)$ , or*
  - (b) *(i)  $s < 2(q - r)$ ,  $\epsilon = 0$ , and there is some  $k \geq 1$  such that there exists a collection of oriented arcs  $\{A_i\}_{i=1}^k$  with the following properties:*
    - (A) *The endpoints of each  $A_i$  are (distinct) essential vertices, neither of which is an endpoint of any other  $A_i$ ,*
    - (B) *There are  $r' \leq r$  vertices of degree greater than 3 which are not the endpoints of any  $A_i$ ,*
    - (C) *There is a collection  $\mathcal{V}$  of degree-3 vertices, with  $|\mathcal{V}| \geq q - r' - k$  such that  $\{A_i\}_{i=1}^k$  is allowable for  $\mathcal{V}$ .*
  - (ii)  *$s < 2(q - r)$ ,  $\epsilon = 1$ , and there is an arc  $A_0$  whose endpoints have no restrictions and whose interior includes a collection  $\mathcal{W}'$  of  $s' \leq q$  distinct vertices of degree 3, and if  $s' < q - r$ , there are arcs  $A_1, \dots, A_k$ , as above whose endpoints are also not vertices in  $\mathcal{W}'$ , and there is another collection of degree-3 vertices,  $\mathcal{W}$ , such that  $\mathcal{W} \cap \mathcal{W}' = \emptyset$ ,  $|\mathcal{W}| \geq q - r' - k - s'$  and  $\{A_i\}_{i=1}^k$  is allowable for  $\mathcal{W}$ , where  $r'$  is as above.*

*then,  $TC(C^n(\Gamma)) = TC(UC^n(\Gamma)) = 2q + 1$ .*

In [5]<sup>1</sup>, Farber proves a similar statement for the spaces  $C^n(\Gamma)$ , showing that  $TC(C^n(\Gamma)) = 2m(\Gamma) + 1$ , for  $m(\Gamma) \geq 1$  and  $n \geq 2m(\Gamma)$ , with an additional assumption that  $\Gamma$  is not homeomorphic to the letter  $Y$  if  $n = 2$ . His proof uses methods which differ from the ones used here, and only address the spaces  $C^n(\Gamma)$  for  $n$  large. Theorem 1.2 in some sense improves this, since it addresses both the

<sup>1</sup>The paper [5] refers to an earlier version of this paper which contained an error.

spaces  $C^n(\Gamma)$  and  $UC^n(\Gamma)$ , and includes some values of  $n$  less than  $2m(\Gamma)$ . Furthermore, Corollary 3.7 determines the topological complexity of both configuration spaces of a tree with no vertices of degree 3 for all values of  $n \geq 1$ , provided the configuration spaces are connected. On the other hand, with  $k \geq 1$  as above, Farber's results determine  $TC(C^n(\Gamma))$  for  $n = 2m, 2m + 1, \dots, 2m + k - 1$ , while our result does not. We discuss in Proposition 3.8 the extent to which the results of Theorem 1.2 are the best one can achieve with the methods used here.

## 2. CONFIGURATION SPACES OF POINTS ON GRAPHS

Consider a graph  $\Gamma$ , where, as above, a graph is a 1-dimensional CW complex. The zero-dimensional cells of  $\Gamma$  are the vertices, and the closures of the 1-dimensional cells are the edges. The degree of a vertex  $v$  is the number of edges which have  $v$  as exactly one of their endpoints plus twice the number of edges which have  $v$  as both endpoints. We will deal exclusively with finite graphs, so that the number of vertices and edges is finite. An essential vertex is a vertex of degree equal to or greater than 3, and  $m(\Gamma)$  is the number of essential vertices in  $\Gamma$ . Let  $C^n(\Gamma)$  be the space of  $n$ -tuples of distinct points in  $\Gamma$ . That is,

$$C^n(\Gamma) = \overbrace{\Gamma \times \cdots \times \Gamma}^{n \text{ times}} - \Delta,$$

where  $\Delta = \{(x_1, \dots, x_n) \in \Gamma \times \cdots \times \Gamma : x_i = x_j \text{ for some } i \neq j\}$ . The space  $C^n(\Gamma)$  will be called the *topological configuration space of  $n$  ordered points on  $\Gamma$* . Similarly, let

$$D^n(\Gamma) = \overbrace{\Gamma \times \cdots \times \Gamma}^{n \text{ times}} - \tilde{\Delta},$$

where  $\tilde{\Delta}$  consists of all products of cells in  $\Gamma \times \cdots \times \Gamma$  whose closures intersect  $\Delta$ . Thus,  $D^n(\Gamma)$  consists of all products of cells  $c_1 \times \cdots \times c_n$  such that  $\bar{c}_i \cap \bar{c}_j = \emptyset$  whenever  $i \neq j$ . In what follows, the word “cell” will always refer to the closure of a cell. A point in  $D^n(\Gamma)$  is then an ordered  $n$ -tuple of points  $(x_1, \dots, x_n)$  in  $\Gamma$  such that there is at least a full open edge between two distinct points  $x_i$  and  $x_j$ . The space  $D^n(\Gamma)$  will be called the *discrete configuration space of  $n$  ordered points on  $\Gamma$* .

There is a free action of the symmetric group  $S_n$  on  $C^n(\Gamma)$  and  $D^n(\Gamma)$  which permutes the coordinates. The quotients of these two spaces under this action are denoted  $UC^n(\Gamma)$  and  $UD^n(\Gamma)$ , and are called the *unordered* topological and discrete configuration spaces. Given a point  $\mathbf{y} = (y_1, \dots, y_n) \in C^n(\Gamma)$ , we may find neighborhoods  $U_i$  in  $\Gamma$  which contain  $y_i$  and satisfy  $U_i \cap U_j = \emptyset$  for  $i \neq j$ . Then,  $U = U_1 \times \cdots \times U_n$  is a neighborhood of  $\mathbf{y}$  in  $C^n(\Gamma)$  and for each  $\alpha \in S_n$ , we have  $\alpha(U) = U_{\alpha(1)} \times \cdots \times U_{\alpha(n)}$ . So, if  $\alpha \neq \alpha' \in S_n$ , then there is some  $i$  such that  $\alpha(i) \neq \alpha'(i)$ , and then  $U_{\alpha(i)} \cap U_{\alpha'(i)} = \emptyset$ , so  $\alpha(U) \cap \alpha'(U) = \emptyset$ . This implies that the quotient map  $C^n(\Gamma) \rightarrow UC^n(\Gamma)$  is a covering space projection (see [10, Proposition 1.40]). If  $\mathbf{y}$  is a point in  $D^n(\Gamma)$ , by letting  $U' = U \cap D^n(\Gamma)$ , we see that the same is true of the quotient map  $D^n(\Gamma) \rightarrow UD^n(\Gamma)$ . The topological and discrete spaces are related by the following theorem due to Abrams:

**Theorem 2.1.** [1],[2] *Let  $\Gamma$  be a graph with at least  $n$  vertices. Suppose*

- (1) *each path between distinct vertices of degree not equal to 2 in  $\Gamma$  contains at least  $n - 1$  edges, and*

- (2) each loop at a vertex in  $\Gamma$  which is not homotopic to a constant map contains at least  $n + 1$  edges.

Then,  $C^n(\Gamma)$  and  $UC^n(\Gamma)$  deformation retract onto  $D^n(\Gamma)$  and  $UD^n(\Gamma)$ , respectively.

A graph which satisfies these conditions is called *sufficiently subdivided for  $n$* . Any graph can be made sufficiently subdivided for any  $n$  by adding enough degree-2 vertices; this has no effect on the topology of the graph or either of the topological configuration spaces. The following relates the connectivity of a graph  $\Gamma$  and the connectivity of the topological configuration spaces:

**Theorem 2.2.** [1] Suppose  $\Gamma$  is a graph with at least one edge or at least  $n + 1$  vertices. Then,

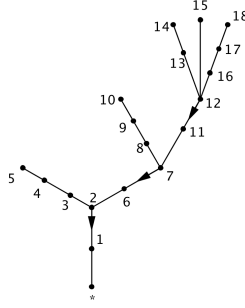
- (1)  $C^n(\Gamma)$  is path-connected if and only if  $\Gamma$  is connected and either
  - (a)  $n = 1$ ,
  - (b)  $n = 2$  and  $\Gamma$  is not homeomorphic to a closed interval, or
  - (c)  $n \geq 3$  and  $\Gamma$  is not homeomorphic to a closed interval or a circle.
- (2)  $UC^n(\Gamma)$  is path-connected if and only if  $\Gamma$  is connected.

It follows that if  $\Gamma$  is sufficiently subdivided, then in Theorem 2.2,  $C^n(\Gamma)$  and  $UC^n(\Gamma)$  can be replaced with  $D^n(\Gamma)$  and  $UD^n(\Gamma)$ . In fact, Abrams also shows that if  $\Gamma$  has at least  $n + 1$  vertices, then  $UD^n(\Gamma)$  is connected if and only if  $\Gamma$  is connected, regardless of subdivision. If  $\Gamma$  has exactly  $n$  vertices, one can easily find examples in which  $\Gamma$  is disconnected, but  $UD^n(\Gamma)$  is a single point (so  $UD^n(\Gamma)$  is connected). There are less trivial examples in [1] in which  $\Gamma$  is connected, while  $D^n(\Gamma)$  is disconnected.

Daniel Farley and Lucas Sabalka have studied extensively the homotopy and homology groups and the cohomology rings of the spaces  $UD^n(\Gamma)$  for a sufficiently subdivided tree  $\Gamma$  using Forman's discrete Morse theory [9]. Recall a tree is a simply connected graph. We summarize some of their results which will be relevant here. From here on,  $\Gamma$  is a tree which is sufficiently subdivided for  $n$ .

First, an ordering on the vertices is constructed as follows. Embed the tree  $\Gamma$  in the plane, and let  $*$  be a vertex of degree 1. Assign  $*$  the number 0, travel away from  $*$ , and number the remaining vertices in order (starting with 1) when they are first encountered. Whenever an essential vertex is encountered, take the leftmost edge, and turn around when a vertex of degree 1 is encountered. For each edge  $e$ , let  $\iota(e)$  and  $\tau(e)$  be the two endpoints of  $e$ , with  $\iota(e) < \tau(e)$ . There is also a notion of *directions* from a vertex  $v \neq *$  of degree  $d$ . These directions are a numbering of the edges incident to  $v$  from 0 to  $d - 1$ , in increasing order clockwise around the vertex, with 0 being the direction on the geodesic segment from  $v$  to  $*$ . An example is given in Figure 1 below. The 0-direction at each essential vertex is marked with an arrow; to avoid clutter, the other directions are not marked on the graph. Note this graph is only sufficiently subdivided for  $n = 2$ , since there is only one edge along the geodesic from vertex 12 to vertex 15.

Recall an arc in  $\Gamma$  is a subspace homeomorphic to a non-trivial closed interval. Farley and Sabalka's notion of directions enables us to define the notion of an allowable collection of oriented arcs. Given a finite collection of oriented arcs  $\{A_i\}_{i=1}^k$  in  $\Gamma$ , and a vertex  $v$  in  $\Gamma$  of degree  $d$ , we will define integers  $\eta_0(v), \eta_1(v), \dots, \eta_{d-1}(v)$  as follows. First, for  $i = 1, 2, \dots, k$  and  $j = 0, 1, \dots, d - 1$ , if  $v$  falls on the arc  $A_i$  and  $A_i$  intersects the interior of the edge  $e_j$  incident to  $v$  in direction  $j$ , let  $\eta_{j,i}(v) = 1$  if


 FIGURE 1. The ordering of the vertices on a tree  $\Gamma$ 

$A_i$  is oriented towards  $v$  on  $e_j$  and  $\eta_{j,i}(v) = -1$  if  $A_i$  is oriented away from  $v$  on  $e_j$ . If  $v$  does not fall on  $A_i$  or if  $A_i$  does not intersect the interior of  $e_j$ , let  $\eta_{j,i}(v) = 0$ . Then, let

$$\eta_j(v) = \sum_{i=1}^k \eta_{j,i}(v).$$

This leads to the following definition:

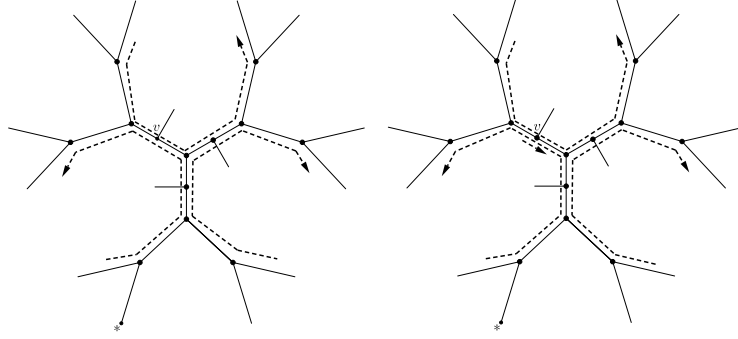
**Definition 2.3.** Suppose  $\mathcal{A} = \{A_i\}_{i=1}^k$  is a collection of oriented arcs in  $\Gamma$  and  $\mathcal{V}$  is a collection of vertices of  $\Gamma$ . The collection  $\mathcal{A}$  is said to be allowable for  $\mathcal{V}$  if every vertex  $v \in \mathcal{V}$  of degree  $d$  has the property that  $v$  is not an endpoint of any  $A_i$ , and at least one of  $\eta_0(v), \eta_1(v), \dots, \eta_{d-1}(v)$  is non-zero.

Intuitively, a collection of oriented arcs is allowable for a given collection  $\mathcal{V}$  of vertices if at each vertex in  $\mathcal{V}$ , there is at least one direction in which the orientations of the arcs don't "cancel out," and no vertex in  $\mathcal{V}$  is an endpoint of any arc. Also, if  $\mathcal{V} = \emptyset$ , then any collection of arcs is allowable for  $\mathcal{V}$ . We will be most interested in the case in which  $\mathcal{V}$  is a subset of the vertices of degree 3. Below is an example of two collections of oriented arcs in a graph  $\Gamma$ . The arcs are shown with a dashed line and orientations indicated with arrows, and have been moved away from  $\Gamma$  so that they are distinguishable in the figure. If  $\mathcal{V}$  is the collection of all essential vertices of  $\Gamma$  (which are all of degree 3), and  $\mathcal{W}$  is the collection of all essential vertices of  $\Gamma$  except the vertex labeled  $v$ , then both collections of arcs are allowable for  $\mathcal{W}$ , but the collection of arcs on the left is not allowable for  $\mathcal{V}$ , while the collection on the right is.

Note that given a collection of oriented arcs  $\mathcal{A} = \{A_i\}_{i=1}^k$  which is allowable for  $\mathcal{V}$  and a vertex  $v \in \mathcal{V}$  of degree  $d$ , if  $j$  is a direction satisfying  $\eta_j(v) \neq 0$ , then there must be some other direction  $j' \neq j$  with  $\eta_{j'}(v) \neq 0$ . Indeed if  $v$  falls on an arc  $A_i$ , then  $v$  must fall on the interior of  $A_i$ , so that  $\eta_{j_1,i}(v) = 1$  for some  $j_1$ , and  $\eta_{j_2,i}(v) = -1$  for some  $j_2 \neq j_1$ . Since  $\Gamma$  is a tree, for all other directions  $j$ , we have  $\eta_{j,i}(v) = 0$ , so  $A_i$  contributes 0 to the sum

$$\sum_{j=0}^{d-1} \sum_{i=1}^k \eta_{j,i}(v) = \sum_{j=0}^{d-1} \eta_j(v).$$

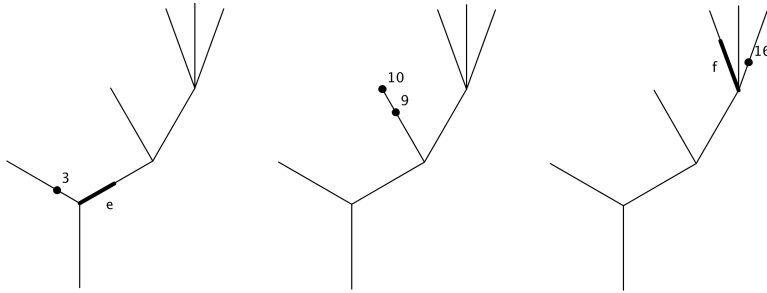
On the other hand, if  $v$  does not fall on  $A_i$ , then  $\eta_{j,i}(v) = 0$  for all  $j$ , so again,  $A_i$  contributes 0 to the sum above. So, we have  $\eta_0(v) + \eta_1(v) + \dots + \eta_{d-1}(v) = 0$ , showing that there cannot be exactly one value  $j$  for which  $\eta_j(v) \neq 0$ . Furthermore,

FIGURE 2. Two collections of arcs in a graph  $\Gamma$ 

for the same reason, if  $\eta_j(v) \neq 0$ , then there must be some direction  $j' \neq j$  such that  $\eta_j(v)$  and  $\eta_{j'}(v)$  have opposite signs.

A cell  $c \in UD^n(\Gamma)$  can be described as a collection of vertices and edges,  $c = \{c_1, \dots, c_n\}$ , where each  $c_i$  is a vertex or an edge, and  $c_i \cap c_j = \emptyset$  for  $i \neq j$ . Note the order in which the vertices and edges appear in  $c$  does not matter. Consider a cell  $c = \{v, c_2, \dots, c_n\}$  containing some vertex  $v \neq *$ . If  $e_v$  is the unique edge in  $\Gamma$  which has  $\tau(e_v) = v$ , and  $\{e_v, c_2, \dots, c_n\}$  is a valid cell in  $UD^n(\Gamma)$ , then  $v$  is said to be *unblocked* in  $c$ . Otherwise,  $v$  is *blocked* in  $c$ . The vertex  $*$  is also said to be blocked in any cell containing it. In other words,  $v$  is unblocked in  $c$  if and only if  $v \neq *$  and  $v$  can be replaced with the edge which contains  $v$  and is on the geodesic segment from  $v$  to  $*$ .

Now, consider a cell  $c = \{e, c_2, \dots, c_n\}$  which contains some edge  $e$ . If there is some vertex  $v$  in  $c$  which has the property that  $\iota(e_v) = \iota(e)$  and  $\iota(e) < v < \tau(e)$ , then  $e$  is said to be *order-disrespecting* in  $c$ . Otherwise, the edge  $e$  is *order-respecting* in  $c$ . An example of 3 different cells in  $UD^2(\Gamma)$  with  $\Gamma$  as in Figure 1 is given below. The vertices/edges which are to be included in a cell  $c$  are labeled; all unlabeled vertices and edges are not included in the cell. In the left cell, the vertex 3 is blocked, since  $e_3$  intersects the edge labeled  $e$ . Also, the edge  $e$  is order-disrespecting since  $\iota(e_3) = 2 = \iota(e)$  and  $\iota(e) = 2 < 3 < \tau(e) = 6$ . In the middle cell, the vertex 10 is blocked, but 9 is unblocked. In the cell on the right, the vertex 16 is blocked, but  $f$  is order-respecting, since although  $\iota(e_{16}) = \iota(f)$ , we have  $\tau(f) = 13 < 16$ .

FIGURE 3. Three different cells in  $UD^2(\Gamma)$

Farley and Sabalka construct a discrete vector field  $W$  and prove the following classification of the critical, collapsible, and redundant cells in  $UD^n(\Gamma)$  with respect to  $W$ . The terms *critical*, *collapsible*, and *redundant* come from discrete Morse theory, but their definitions will not be needed here.

**Theorem 2.4.** [7] *A cell  $c \in UD^n(\Gamma)$  is critical if and only if each vertex in  $c$  is blocked and each edge in  $c$  is order-disrespecting. A cell  $c$  is collapsible if and only if it contains some order-respecting edge  $e$  with the property that any unblocked vertex  $v$  in  $c$  satisfies  $v > \tau(e)$ . All other cells are redundant.*

Since we will be focused primarily on critical cells, we describe a procedure to construct a critical  $k$ -cell in  $UD^n(\Gamma)$ . First, notice that if  $c$  is any  $k$ -cell, then  $c$  must consist of  $k$  edges and  $n - k$  vertices. If  $c$  is to be critical, each edge must be order-disrespecting, and each vertex must be blocked. Consider an edge  $e$  in  $c$ . If  $e$  is to be order-disrespecting, then  $\iota(e)$  must be an essential vertex, or else there would be no possible vertex  $v$  that could satisfy  $\iota(e_v) = \iota(e)$  other than  $v = \tau(e)$ , but  $\tau(e)$  cannot be included in a cell which contains  $e$  (by this, we mean that  $\tau(e)$  cannot appear as a vertex in the list of vertices and edges which define  $c$ ). Furthermore, the direction  $d$  from  $\iota(e)$  on which  $e$  falls must be at least 2, and the direction  $d'$  from  $\iota(e)$  on which  $v$  falls must satisfy  $0 < d' < d$ . Note also that if  $v$  causes  $e$  to be order-disrespecting, then  $v$  is automatically blocked. This also implies that if there is a critical  $k$ -cell in  $UD^n(\Gamma)$ , then  $k \leq m(\Gamma)$  and  $n \geq 2k$ . The remaining  $n - 2k$  vertices of  $c$  can be easily chosen so that they are blocked in  $c$ . The cell on the left in Figure 3 is a critical 1-cell in  $UD^2(\Gamma)$ . Below is an example of a critical 3-cell in  $UD^8(\Gamma)$ . Note that more vertices of degree 2 must be added to the tree  $\Gamma$  above so that it is sufficiently subdivided for  $n = 8$ . In this example and what follows, we make no indication of the total number of vertices in a sufficiently subdivided tree.

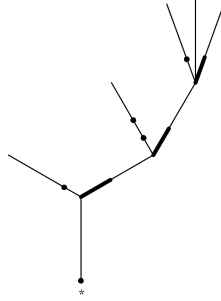


FIGURE 4. A critical 3-cell  $c$  in  $UD^8(\Gamma)$

The discussion above is similar to the proof of the following theorem:

**Theorem 2.5.** [7] *Let  $\Gamma$  be a tree, and let  $k = \min \left\{ \left\lfloor \frac{n}{2} \right\rfloor, m(\Gamma) \right\}$ . If  $c$  is any critical cell in  $UD^n(\Gamma)$ , then  $\dim(c) \leq k$ . Furthermore,  $UD^n(\Gamma)$  deformation retracts onto the space  $(UD^n(\Gamma))'_k$ , which consists of the  $k$ -skeleton of  $UD^n(\Gamma)$  with the redundant  $k$ -cells removed.*

The second statement of this theorem follows from results from discrete Morse theory. Now, before discussing the cohomology ring  $H^*(UD^n(\Gamma); \mathbb{Z})$ , we describe

the equivalence relation on cells given in [8]. Given two cells  $c, c' \in UD^n(\Gamma)$ , define “ $\sim$ ” by  $c \sim c'$  if and only if  $c$  and  $c'$  share the same edges (so in particular  $c$  and  $c'$  are of the same dimension), and if  $E$  is the set of edges in  $c$  (and in  $c'$ ), and  $C = \Gamma - E$ , then for every connected component  $C$  of  $C$ , the number of vertices of  $c$  in  $C$  equals the number of vertices of  $c'$  in  $C$ . Here and in what follows, we use  $E$  to denote both the set of edges and the union of the edges in  $E$ . Context should make the desired interpretation of  $E$  clear. Let  $[c]$  denote the equivalence class of  $c$ . Now, given two equivalence classes  $[c]$ ,  $[d]$ , write  $[d] \leq [c]$  if there are representatives  $c \in [c]$  and  $d \in [d]$  such that  $d$  is obtained from  $c$  by removing some (possibly zero) edges of  $c$  and replacing each of these edges with one of its endpoints. Farley and Sabalka show the following:

**Lemma 2.6.** [8] *The relation “ $\leq$ ” is a well-defined partial order that has the additional property that for any  $k$ -cell  $c \in UD^n(\Gamma)$ , there is a unique collection of equivalence classes of 1-cells  $\{[c_1], [c_2], \dots, [c_k]\}$  having  $[c]$  as its least upper bound.*

Farley and Sabalka also introduced the idea of a “cloud diagram” to represent an equivalence class. These diagrams consist of a collection  $E$  of edges and an indication of the number of vertices in each connected component of  $\Gamma - E$ . The components of  $\Gamma - E$  are called *clouds*. If  $f(C)$  is the number of vertices in a cloud  $C$ , then

$$\sum_{\text{Clouds } C} f(C) = n - |E|.$$

We will sometimes call the number of vertices in a cloud  $C$  the *value* of  $C$ . Cloud diagrams also provide a convenient way to determine if  $[d] \leq [c]$ . If this is the case, then the set of edges in the diagram for  $[d]$  must be a subset of the set of edges in the diagram for  $[c]$ , which implies that each cloud in the diagram for  $[c]$  must be contained in some cloud in the diagram for  $[d]$ . For each cloud  $D$  in the diagram for  $[d]$ , the number of edges of  $[c]$  which are contained in  $D$  plus the sum of the values of the clouds of  $[c]$  which are contained in  $D$  must equal the value of  $D$ . In particular, if the set of edges in  $[c]$  equals the set of edges in  $[d]$ , then the diagrams for  $[c]$  and  $[d]$  have the same clouds, and  $[c]$  and  $[d]$  are comparable iff the values of each cloud are the same in both diagrams, in which case  $[c] = [d]$ . Below is an example. The cloud diagram on the left represents the class  $[c]$ , with  $c$  as in Figure 4. The middle diagram is a cloud diagram for a class  $[d]$  with  $[d] \leq [c]$  and the diagram on the right is a cloud diagram for a class  $[d']$  which is comparable to neither  $[c]$  nor  $[d]$ .

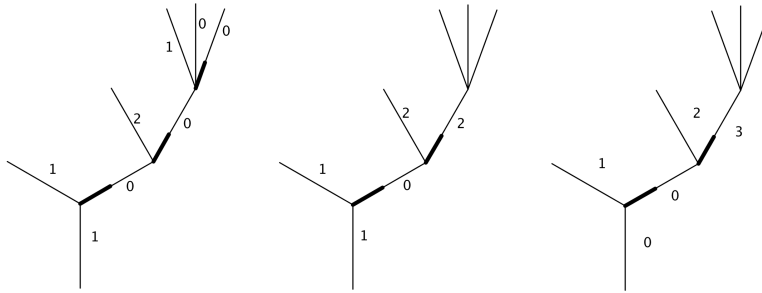


FIGURE 5. Cloud Diagrams for classes  $[c]$ , with  $c$  as in Figure 4 (left),  $[d]$  (middle), and  $[d']$  (right)



For each cell  $c \in UD^n(\Gamma)$ , define a cellular cochain  $\phi_{[c]} \in C^*(UD^n(\Gamma); \mathbb{Z})$  by

$$\phi_{[c]}(c') = \begin{cases} 1, & \text{if } c' \sim c \\ 0, & \text{otherwise.} \end{cases}$$

If there is a (unique) critical cell in  $[c]$ , then  $\phi_{[c]}$  is called a critical cocycle. The following gives information about the product structure of the integral cohomology ring  $H^*(UD^n(\Gamma))$  :

**Theorem 2.7.** [8],[6]

- (1) *The collection of critical cocycles forms a basis for  $H^*(UD^n(\Gamma))$ .*
- (2) *For any critical cell  $c$ , we have  $\phi_{[c]}^2 = 0$ .*
- (3) *If  $c$  is a critical  $k$ -cell, and  $\{[c_1], \dots, [c_k]\}$  is the unique collection of equivalence classes of 1-cells with  $[c]$  as its least upper bound, arranged such that  $\iota(e_i) < \iota(e_{i+1})$ , where  $e_j$  is the unique edge in  $[c_j]$ , then*

$$\phi_{[c_1]} \cdots \phi_{[c_k]} = \phi_{[c]}.$$

- (4) *If  $\{[c_1], \dots, [c_j]\}$  is any collection of equivalence classes of cells with no upper bound, then*

$$\phi_{[c_1]} \cdots \phi_{[c_j]} = 0$$

It follows from the universal coefficient theorem that the same statements hold true for rational cohomology, where we identify  $\phi_{[c]} \in H^*(UD^n(\Gamma); \mathbb{Z})$  with the corresponding class in  $H^*(UD^n(\Gamma); \mathbb{Q})$ . More information is given about integral cohomology in [6], but the above is sufficient for what follows.

### 3. MOTION PLANNING OF CONFIGURATION SPACES OF TREES

Before proving Theorem 1.2, we first mention some of the tools for determining the topological complexity of any space.

**Theorem 3.1.** [3]  *$TC(X)$  is homotopy-invariant. That is, if  $X$  and  $Y$  are homotopic, then  $TC(X) = TC(Y)$ .*

Since we assume  $\Gamma$  is sufficiently subdivided, the spaces  $C^n(\Gamma)$  and  $UC^n(\Gamma)$  are homotopic to  $D^n(\Gamma)$  and  $UD^n(\Gamma)$ , respectively, by Theorem 2.1, so Theorem 3.1 allows us to work with  $D^n(\Gamma)$  and  $UD^n(\Gamma)$  to determine  $TC(C^n(\Gamma))$  and  $TC(UC^n(\Gamma))$ . The next theorem gives an upper bound for  $TC(X)$  based on the dimension of  $X$  :

**Theorem 3.2.** [3] *Let  $X$  be any CW complex. Then, we have the upper bound  $TC(X) \leq 2 \cdot \dim(X) + 1$ .*

Finally, there is a cohomological lower bound for  $TC(X)$ . Before stating the theorem, we introduce some definitions. Let  $\mathbf{k}$  be a field, and consider the cup product

$$\smile: H^*(X; \mathbf{k}) \otimes H^*(X; \mathbf{k}) \rightarrow H^*(X; \mathbf{k}).$$

Let  $Z(X) \subset H^*(X; \mathbf{k}) \otimes H^*(X; \mathbf{k})$  be the kernel of this homomorphism, called the ideal of *zero-divisors* of  $H^*(X; \mathbf{k})$ . The tensor product  $H^*(X; \mathbf{k}) \otimes H^*(X; \mathbf{k})$  has a multiplication given by  $(\alpha \otimes \beta)(\alpha' \otimes \beta') = (-1)^{|\alpha'| \cdot |\beta|} \alpha \alpha' \otimes \beta \beta'$ , where  $|x| = j$  if  $x \in H^j(X; \mathbf{k})$ . The *zero-divisors-cup-length* of  $H^*(X; \mathbf{k})$  is the largest  $i$  such there are elements  $a_1, \dots, a_i \in Z(X)$  with  $a_1 \cdots a_i \neq 0$ .

**Theorem 3.3.** [3]  *$TC(X)$  is greater than the zero-divisors-cup-length of  $H^*(X; \mathbf{k})$ .*

Now, we establish the upper bounds in Theorem 1.2:

**Lemma 3.4.** *Let  $k = \min \left\{ \left\lfloor \frac{n}{2} \right\rfloor, m(\Gamma) \right\}$ . Then,  $TC(UD^n(\Gamma)) \leq 2k + 1$ .*

*Proof.* This is immediate from the second statement of Theorem 2.5 and Theorems 3.1 and 3.2, but we will give a (fairly) explicit motion planning algorithm which realizes this upper bound. This algorithm is similar to the one given by Farber in [4]. Let  $a_1, \dots, a_n$  be the first  $n$  vertices in a sufficiently subdivided tree  $\Gamma$  (so  $a_1 = *$ ). The fact that  $\Gamma$  is sufficiently subdivided implies that only possibly  $a_n$  is essential. Consider a point  $\mathbf{x} = \{x_1, \dots, x_n\} \in UD^n(\Gamma)$ . If  $x_i$  falls on a vertex  $v$ , let  $f(x_i)$  be the number assigned to  $v$  in the ordering above. If  $x_i$  falls on the interior of an edge  $e$ , let  $f(x_i)$  be the number assigned to  $\iota(e)$ . Since the order in which the  $x_i$  appear in  $\mathbf{x}$  is irrelevant, we may assume without loss of generality that  $f(x_i) < f(x_{i+1})$  for all  $i$ . Define a map  $\sigma_{\mathbf{x}} : [0, 1] \rightarrow UD^n(\Gamma)$  as follows. During the interval  $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ ,  $\sigma_{\mathbf{x}}$  is the path which moves  $x_i$  along the geodesic to  $a_i$  at constant speed, and keeps all other  $x_j$  fixed. The choice of ordering of the vertices makes this a valid path in  $UD^n(\Gamma)$  (i.e. there is at least a full open edge between any two components of  $\sigma_{\mathbf{x}}(t)$  at any given time  $t$ ). Each  $\sigma_{\mathbf{x}}$  is clearly continuous. Define the section  $s : UD^n(\Gamma) \times UD^n(\Gamma) \rightarrow P(UD^n(\Gamma))$  by  $s(\mathbf{x}, \mathbf{y}) = \sigma_{\mathbf{x}} \overline{\sigma_{\mathbf{y}}}$ , the path  $\sigma_{\mathbf{x}}$  followed by the reverse of  $\sigma_{\mathbf{y}}$ .

This is not continuous on  $UD^n(\Gamma) \times UD^n(\Gamma)$ . If some  $x_i$  (or  $y_i$ ) falls on the interior of an edge  $e$ , a slight perturbation of  $x_i$  (or  $y_i$ ) may cause it to fall on  $\tau(e)$ , which can alter the numbering of the elements in  $\mathbf{x}$  (or  $\mathbf{y}$ ), which can lead to a very different path if  $\iota(e)$  is essential. So, we wish to examine the sets on which  $s$  is continuous. For a collection  $E$  of edges, let  $S_E$  be the set of points  $\mathbf{x} = \{x_1, \dots, x_n\} \in UD^n(\Gamma)$  with the property that the interior of each edge  $e \in E$  contains (exactly) one  $x_i$  and no  $x_j$  falls on the interior of any edge not in  $E$  (so such an  $x_j$  falls on a vertex which is not the endpoint of any  $e \in E$ ). The function  $s$  is continuous on each  $S_E \times S_{E'}$ . Now, let

$$S_i = \bigcup_{|E|=i} S_E.$$

If  $|E| = |E'|$  and  $E \neq E'$ , then a sequence of points in  $S_E$  cannot converge to a point in  $S_{E'}$ , so that  $\overline{S_E} \cap S_{E'} = \emptyset$ , and similarly  $S_E \cap \overline{S_{E'}} = \emptyset$ . In other words,  $S_i$  is a topological disjoint union of the sets  $S_E$ , and then for each fixed  $i, j$ , the set  $S_i \times S_j$  is a topological disjoint union of sets on which  $s$  is continuous, so  $s$  is continuous on  $S_i \times S_j$ . Now, a sequence of points in  $S_i$  may converge to a point in  $S_{i'}$  for some  $i' < i$ , but no sequence of points in  $S_i$  can converge to a point in  $S_{i'}$  if  $i' > i$ , so the sets

$$U_l = \bigcup_{i+j=l} S_i \times S_j$$

are again topological disjoint unions of sets on which  $s$  is continuous, so  $s$  is continuous on  $U_l$ .

The sets  $U_0, \dots, U_{2k}$  cover  $(UD^n(\Gamma))'_k \times (UD^n(\Gamma))'_k$ , since at most  $k$  points can fall on the interior of an edge in either factor (see Theorem 2.5). They are not necessarily open, but each  $U_l$  can be replaced with an open set  $U'_l$  which allows each  $x_i$  which falls on a vertex  $v$  (and  $x_i$  appears in a point  $\mathbf{x}$  in the first component of  $U_l$ ) to vary slightly away from  $v$  (while keeping the point  $\mathbf{x}$  in  $UD^n(\Gamma)$ ), and

defining  $\sigma_{\mathbf{x},l}$  which is as above, except each of these  $x_i$  is given the number for  $v$ . This is well-defined, since each  $v$  does not fall on any of the  $l$  edges whose interiors are occupied by some  $x_j$  in  $\mathbf{x}$ , so if  $x_i$  falls on  $v$ , a small perturbation of  $x_i$  will not cause it to fall on the interior of any of those  $l$  edges. Similar modifications are made in the second component. Define  $s'_l : U'_l \rightarrow P(UD^n(\Gamma))$  by

$$s'_l(\mathbf{x}, \mathbf{y}) = \sigma_{\mathbf{x},l} \bar{\sigma}_{\mathbf{y},l}$$

This is continuous.

If  $H : [0, 1] \times UD^n(\Gamma) \rightarrow UD^n(\Gamma)$  is a deformation retraction from  $UD^n(\Gamma)$  to  $(UD^n(\Gamma))'_k$  with  $H(0, -)$  equalling the identity map, then,  $H_{\mathbf{x}} := H(-, \mathbf{x})$  is a path from a point  $\mathbf{x} \in UD^n(\Gamma)$  to some point in  $(UD^n(\Gamma))'_k$ , which varies continuously with  $\mathbf{x}$ . If  $V_l = (H(1, -) \times H(1, -))^{-1}(U'_l)$ , then  $\{V_0, \dots, V_{2k}\}$  is an open cover of  $UD^n(\Gamma) \times UD^n(\Gamma)$ , and the section  $s_l : V_l \rightarrow P(UD^n(\Gamma))$  given by

$$s_l(\mathbf{x}, \mathbf{y}) = H_{\mathbf{x}} s'_l(\mathbf{x}, \mathbf{y}) \bar{H}_{\mathbf{y}}$$

is continuous on each  $V_l$ , for  $l = 0, \dots, 2k$ , completing the proof.  $\square$

An additional step can be added to this algorithm to give the same upper bound for the ordered configuration spaces. Again, this approach is essentially the one described in [4].

**Corollary 3.5.** *Let  $\Gamma$  be a tree with  $m(\Gamma) \geq 1$ , and let  $k = \min\{\lfloor \frac{n}{2} \rfloor, m(\Gamma)\}$ . Then,  $TC(D^n(\Gamma)) \leq 2k + 1$ .*

*Proof.* Let  $a_1, \dots, a_n$  be the first  $n$  vertices of  $\Gamma$ , and let  $\mathbf{a} = \{a_1, \dots, a_n\} \in UD^n(\Gamma)$ . For  $l = 0, \dots, 2k$ , let  $V_l$ ,  $\sigma_{\mathbf{x},l}$  and  $H_{\mathbf{x}}$  be as in the proof of Lemma 3.4. Let  $p_1 : UD^n(\Gamma) \times UD^n(\Gamma) \rightarrow UD^n(\Gamma)$  be the projection of the first component, and let  $V_l^1 = p_1(V_l)$ , so that if  $\mathbf{x} \in V_l^1$ , then  $H_{\mathbf{x}} \sigma_{\mathbf{x},l}$  is a path from  $\mathbf{x}$  to  $\mathbf{a}$ , and this path varies continuously as  $\mathbf{x}$  varies in  $V_l^1$ . If  $\pi$  is the covering space projection  $\pi : D^n(\Gamma) \rightarrow UD^n(\Gamma)$ , and  $\tilde{\mathbf{x}} \in \pi^{-1}(\mathbf{x}) \subset D^n(\Gamma)$ , with  $\mathbf{x} \in V_l^1$ , let  $\tilde{\sigma}_{\tilde{\mathbf{x}},l}$  be the unique lift of  $H_{\mathbf{x}} \sigma_{\mathbf{x},l}$  which satisfies  $\tilde{\sigma}_{\tilde{\mathbf{x}},l}(0) = \tilde{\mathbf{x}}$ . This is a path from  $\tilde{\mathbf{x}}$  to some point  $\tilde{\mathbf{a}} \in \pi^{-1}(\mathbf{a})$  which varies continuously as  $\tilde{\mathbf{x}}$  varies in  $\tilde{V}_l^1 = \pi^{-1}(V_l^1)$ . Define  $\tilde{\sigma}_{\tilde{\mathbf{y}},l}$  similarly if  $\tilde{\mathbf{y}} \in \pi^{-1}(\mathbf{y})$  for some  $\mathbf{y}$  in the second component of  $V_l$ .

Now, given  $\tilde{\mathbf{a}}, \tilde{\mathbf{a}}' \in \pi^{-1}(\mathbf{a})$ , let  $\rho_{\tilde{\mathbf{a}}, \tilde{\mathbf{a}}'}$  be any path from  $\tilde{\mathbf{a}}$  to  $\tilde{\mathbf{a}}'$ . Such a path exists by Theorem 2.2 since  $\Gamma$  has at least one essential vertex. The function

$$r : \pi^{-1}(\mathbf{a}) \times \pi^{-1}(\mathbf{a}) \rightarrow P(D^n(\Gamma))$$

given by  $r(\tilde{\mathbf{a}}, \tilde{\mathbf{a}}') = \rho_{\tilde{\mathbf{a}}, \tilde{\mathbf{a}}'}$  is continuous since the domain is a discrete space. For each  $l = 0, \dots, 2k$ , let  $\tilde{V}_l = (\pi \times \pi)^{-1}(V_l)$ , and define the section  $\tilde{s}_l : \tilde{V}_l \rightarrow P(D^n(\Gamma))$  by

$$\tilde{s}_l(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \tilde{\sigma}_{\tilde{\mathbf{x}},l} r(\tilde{\sigma}_{\tilde{\mathbf{x}},l}(1), \tilde{\sigma}_{\tilde{\mathbf{y}},l}(1)) \overline{\tilde{\sigma}_{\tilde{\mathbf{y}},l}}.$$

This is continuous on  $\tilde{V}_l$ , and  $\{\tilde{V}_0, \dots, \tilde{V}_{2k}\}$  is an open cover of  $D^n(\Gamma) \times D^n(\Gamma)$ , completing the proof.  $\square$

In establishing the lower bound, the following will be useful:

**Lemma 3.6.** *Suppose  $m(\Gamma) \geq 1$ ,  $UD^n(\Gamma)$  has the homotopy type of a  $k$ -dimensional CW complex, and  $\Phi$  and  $\Psi$  are critical  $k$ -cells in  $UD^n(\Gamma)$ . If  $\{[c_1], \dots, [c_k]\}$  and  $\{[d_1], \dots, [d_k]\}$  are the unique collections of equivalence classes of 1-cells having least upper bounds  $[\Phi]$  and  $[\Psi]$ , respectively, and  $[c_i] \neq [d_j]$  for all  $i, j$ , then  $TC(UD^n(\Gamma))$  and  $TC(D^n(\Gamma))$  are greater than  $2k$ .*

*Proof.* If necessary, rearrange the equivalence classes in  $\{[c_1], \dots, [c_k]\}$  so that  $\iota(e_i) < \iota(e_{i+1})$ , where  $e_i$  is the unique edge in  $[c_i]$ . Arrange the classes in  $\{[d_1], \dots, [d_k]\}$  similarly. For  $i = 1, \dots, k$ , let  $\gamma_i = \phi_{[c_i]} \in H^*(UD^n(\Gamma); \mathbb{Q})$  and  $\delta_i = \phi_{[d_i]} \in H^*(UD^n(\Gamma); \mathbb{Q})$ , and consider the zero divisors

$$\bar{\gamma}_i = \gamma_i \otimes 1 - 1 \otimes \gamma_i, \quad \bar{\delta}_j = \delta_j \otimes 1 - 1 \otimes \delta_j$$

in  $Z(UD^n(\Gamma)) \subset H^*(UD^n(\Gamma); \mathbb{Q}) \otimes H^*(UD^n(\Gamma); \mathbb{Q})$  and their product

$$\begin{aligned} \left( \prod_{i=1}^k \bar{\gamma}_i \right) \left( \prod_{j=1}^k \bar{\delta}_j \right) &= (-1)^k (\gamma_1 \cdots \gamma_k \otimes \delta_1 \cdots \delta_k) + (-1)^{k^3} (\delta_1 \cdots \delta_k \otimes \gamma_1 \cdots \gamma_k) + \text{other terms} \\ &= \pm (\phi_{[\Phi]} \otimes \phi_{[\Psi]} + \phi_{[\Psi]} \otimes \phi_{[\Phi]}) + \text{other terms}. \end{aligned}$$

Since  $UD^n(\Gamma)$  has the homotopy type of a  $k$ -dimensional complex, all products of more than  $k$  1-dimensional classes in  $H^*(UD^n(\Gamma))$  are zero, so, the *other terms* must be of the form  $\alpha \otimes \beta$  where  $\alpha$  and  $\beta$  are both degree- $k$  monomials in  $\gamma_1, \dots, \gamma_k, \delta_1, \dots, \delta_k$ . For such an  $\alpha = \gamma_{i_1} \cdots \gamma_{i_p} \delta_{i_{p+1}} \cdots \delta_{i_k}$ , if the collection  $\{[c_{i_1}], \dots, [c_{i_p}], [d_{i_{p+1}}], \dots, [d_{i_k}]\}$  has a least upper bound  $[\omega]$ , then there is some critical  $k$ -cell  $\omega \in [\omega]$ , and  $\alpha = \phi_{[\omega]}$ , a basis element in  $H^*(UD^n(\Gamma))$ . Otherwise,  $\alpha = 0$ . Furthermore, since  $\alpha$  contains at least one  $\gamma_i$  and at least one  $\delta_j$ , and  $[c_i] \neq [d_j]$ , the collection  $\{[c_{i_1}], \dots, [c_{i_p}], [d_{i_{p+1}}], \dots, [d_{i_k}]\}$  is neither  $\{[c_1], \dots, [c_k]\}$  nor  $\{[d_1], \dots, [d_k]\}$ . The fact that the collections which have  $[\Phi]$  and  $[\Psi]$  as their least upper bounds are unique implies that  $[\omega] \neq [\Phi]$  and  $[\omega] \neq [\Psi]$ , so  $\alpha \neq \phi_{[\Phi]}$  and  $\alpha \neq \phi_{[\Psi]}$ . Similarly,  $\beta$  is either zero or a basis element not equal to  $\phi_{[\Phi]}$  or  $\phi_{[\Psi]}$ .

So, the sum above is a sum of basis elements of  $H^*(UD^n(\Gamma)) \otimes H^*(UD^n(\Gamma))$ , with neither  $\phi_{[\Phi]} \otimes \phi_{[\Psi]}$  nor  $\phi_{[\Psi]} \otimes \phi_{[\Phi]}$  appearing in *other terms*, and since  $\Phi$  and  $\Psi$  are critical cells,  $\phi_{[\Phi]} \otimes \phi_{[\Psi]} + \phi_{[\Psi]} \otimes \phi_{[\Phi]} \neq 0$ , so the entire sum is non-zero. Thus, we have a non-zero product of  $2k$  zero-divisors, and Theorem 3.3 establishes that  $TC(UD^n(\Gamma)) \geq 2k + 1$ , as desired.

The statement for  $TC(D^n(\Gamma))$  follows from the fact that that map  $\pi^* : H^*(UD^n(\Gamma)) \rightarrow H^*(D^n(\Gamma))$  induced by the covering space projection  $\pi : D^n(\Gamma) \rightarrow UD^n(\Gamma)$  is injective (see [10, Proposition 3G.1], which states that any  $N$ -sheeted covering space projection given by a group action induces an injection in cohomology with coefficients in a field of characteristic 0).  $\square$

Note that it is necessary that  $[c_i] \neq [d_j]$  for all  $i, j$ . If  $[c_i] = [d_j]$  for some  $i, j$ , then  $\gamma_i = \delta_j$ , so the product above contains the product

$$\begin{aligned} \bar{\gamma}_i^2 &= (\gamma_i \otimes 1 - 1 \otimes \gamma_i)(\gamma_i \otimes 1 - 1 \otimes \gamma_i) \\ &= (\gamma_i \otimes 1)^2 - (\gamma_i \otimes 1)(1 \otimes \gamma_i) - (1 \otimes \gamma_i)(\gamma_i \otimes 1) + (1 \otimes \gamma_i)^2 \\ &= \gamma_i^2 \otimes 1 - (-1)^0 (\gamma_i \otimes \gamma_i) - (-1)^1 (\gamma_i \otimes \gamma_i) + 1 \otimes \gamma_i^2 \\ &= 0. \end{aligned}$$

We now prove Theorem 1.2.

*Proof of Theorem 1.2.* Assume without loss of generality that  $\Gamma$  is sufficiently subdivided for  $n$ , so that the topological and discrete configuration spaces have the same homotopy type, and hence, the same topological complexity. We will prove the results for the discrete configuration spaces. The upper bounds,  $TC(UD^n(\Gamma)) \leq 2m + 1$ ,  $TC(D^n(\Gamma)) \leq 2m + 1$  in statement 1 and  $TC(UD^n(\Gamma)) \leq 2q + 1$ ,  $TC(D^n(\Gamma)) \leq 2q + 1$ , in statement 2 are given in Lemma 3.4 and Corollary 3.5. To establish

the lower bounds, we will construct two  $m$ -cells for statement 1 and two  $q$ -cells for statement 2 that satisfy the conditions of Lemma 3.6.

For statement 1, first note that Theorem 2.5 implies  $UD^n(\Gamma)$  is homotopic to an  $m$ -dimensional CW complex. Let  $v_1, \dots, v_m$  be (in order) the essential vertices of  $\Gamma$  and let  $A_1, \dots, A_k$  be a collection of oriented arcs which is allowable for the set of vertices of degree 3. For each arc  $A_l$ , let  $a_l$  and  $b_l$  be the initial and terminal points of  $A_l$ , with respect to the orientation of  $A_l$ . The endpoints need not be vertices of  $\Gamma$ .

We will construct an  $m$ -cell  $\Phi$  as follows. At each essential vertex  $v_i$ , let  $e_i$  be the edge in direction 2 from  $v_i$  (so that  $\iota(e_i) = v_i$ ), let  $f_i$  be the edge in direction 1 from  $v_i$ , and let  $u_i = \tau(f_i)$ . The labeling of the vertices forces  $v_i = \iota(e_i) < u_i < \tau(e_i)$ , so that  $e_i$  is order-disrespecting in any cell containing  $u_i$ . Furthermore,  $u_i$  is blocked in any cell containing  $e_i$ . Add each edge  $e_i$  and each vertex  $u_i$  to  $\Phi$ . The edges  $e_i$  determine a system of clouds of  $\Gamma$ .

Now, if  $n > 2m$  (which must be the case, by assumption, if there are any vertices of degree 3), we must have  $n \geq 3$ , so since  $\Gamma$  is sufficiently subdivided, if the endpoint  $a_l$  falls on an edge  $e_i$ , we may shrink or enlarge  $A_l$  slightly so that the new initial point falls just beyond  $\tau(e_i)$ , without changing the fact that the collection  $\{A_i\}_{i=1}^k$  is allowable for the collection of degree-3 vertices. So, we may assume that each endpoint  $a_l$  falls in one of the clouds of  $\Gamma$  determined by the edges  $e_i$ . Then, inductively, for  $l = 1, \dots, k$ , let  $w_l$  be the minimal vertex in the cloud containing  $a_l$  which we have not already included in  $\Phi$ , and add this vertex to  $\Phi$ . Then,  $w_1, \dots, w_k$  are all blocked in  $\Phi$ . Finally, let  $w_{k+1}, \dots, w_{n-2m}$  be the first  $n - 2m - k$  vertices of  $\Gamma$  which are not already in  $\Phi$ , and add them to  $\Phi$  (so they are all blocked).

So, we have

$$\Phi = \{e_1, \dots, e_m, u_1, \dots, u_m, w_1, \dots, w_{n-2m}\},$$

where the vertices  $w_i$  do not appear if  $n = 2m$ . The cell  $\Phi$  is critical. Now, we will construct a critical  $m$ -cell  $\Psi$  similarly. If  $v_i$  is a vertex of degree 3, let  $e'_i = e_i$  and  $u'_i = u_i$ . If  $v_i$  is a vertex of degree greater than 3, let  $e'_i$  be the edge in direction 3 from  $v_i$ , let  $f'_i$  be the edge in direction 2, and let  $u'_i = \tau(f'_i)$ , so  $u'_i$  is blocked by  $e'_i$ . As above, we have  $v'_i = \iota(e'_i) < u'_i < \tau(e'_i)$ , so  $e'_i$  is order-disrespecting in any cell containing  $e'_i$ . Add the edges  $e'_i$  and vertices  $u'_i$  to the cell  $\Psi$ . Similar to the argument above, if  $n > 2m$ , we may assume each endpoint  $b_l$  falls in one of the clouds of  $\Gamma$  determined by the edges  $e'_i$ . Inductively for  $l = 1, \dots, k$ , let  $w'_l$  be the minimal vertex in the cloud containing  $b_l$  which we have not already included in  $\Psi$ , and add this vertex to  $\Psi$ . Let  $w'_{k+1}, \dots, w'_{n-2m}$  be the first  $n - 2m - k$  vertices of  $\Gamma$  which are not already in  $\Psi$ , and add them in, so now we have

$$\Psi = \{e'_1, \dots, e'_m, u'_1, \dots, u'_m, w'_1, \dots, w'_{n-2m}\},$$

where the vertices  $w'_i$  do not appear if  $n = 2m$ . The cell  $\Psi$  is critical. An example is given below for  $m = 11$ ,  $k = 3$ , and  $n = 27$ . The figure on the left is a tree  $\Gamma$  with 3 oriented arcs which are allowable for the collection of degree-3 vertices. The orientations of the arcs are indicated with arrows.

Now, let  $\{[c_1], \dots, [c_m]\}$  be the unique collection of equivalence classes of 1-cells having  $[\Phi]$  as its least upper bound. The equivalence class  $[c_i]$  can be represented using a cloud diagram having a single edge (the edge  $e_i$ ) in direction 2 from the essential vertex  $v_i$ , whose degree is  $t_i$ , and  $t_i$  clouds. Likewise, let  $\{[d_1], \dots, [d_m]\}$

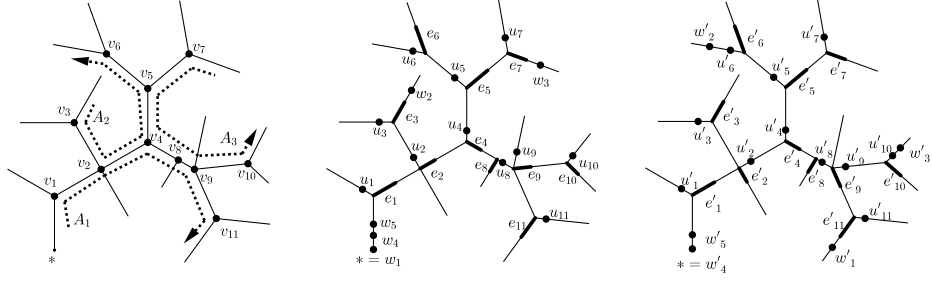


FIGURE 6. A graph  $\Gamma$  (left) and the critical cells  $\Phi$  and  $\Psi$  in  $UD^{27}(\Gamma)$  (center and right)

be the unique collection of equivalence classes of 1-cells having  $[\Psi]$  as its least upper bound. The equivalence class  $[d_i]$  can be represented using a cloud diagram having a single edge (the edge  $e'_i$ ) and  $t_i$  clouds. If  $t_i \geq 4$ , then  $[c_i] \neq [d_i]$ , since the edges  $e_i$  and  $e'_i$  differ.

If  $t_i = 3$ , then, the cloud diagram for  $[c_i]$  will contain the single edge  $e_i$  and three clouds  $C_0, C_1, C_2$ , in the 0, 1, and 2 directions from  $v_i$ . The cloud diagram for  $[d_i]$  will have the same edge and clouds. Suppose  $f(C)$  is the value assigned to cloud  $C$  in the diagram for  $[c_i]$  and  $g(C)$  is the value assigned to cloud  $C$  in the diagram for  $[d_i]$ . For  $\delta = 0, 1, 2$ , let  $m_\delta$  be the number of essential vertices in  $C_\delta$ , and let  $W_\delta = |\{p : w_p \in C_\delta\}|$  and  $W'_\delta = |\{p : w'_p \in C_\delta\}|$ . Then  $f(C_\delta) = 2m_\delta + W_\delta + \lambda_\delta$  and  $g(C_\delta) = 2m_\delta + W'_\delta + \lambda_\delta$ , where  $\lambda_0 = \lambda_2 = 0$  and  $\lambda_1 = 1$ .

Suppose the vertex  $v_i$  falls on the interior of an arc  $A_l$  (by assumption there is at least one such arc). For each direction  $\delta$ , if the arc  $A_l$  is oriented towards  $v_i$  in the direction  $\delta$ , then  $a_l$  must fall in  $C_\delta$ , and  $b_l$  must fall in a different cloud, so the vertex  $w_l$  contributes 1 to  $W_\delta$ , but  $w'_l$  contributes 0 to  $W'_\delta$ . On the other hand, if  $A_l$  is oriented away from  $v_i$  in the direction  $\delta$ , then  $b_l$  must fall in  $C_\delta$ , and  $a_l$  must fall in a different cloud so the vertex  $w'_l$  contributes 1 to  $W'_\delta$ , but  $w_l$  contributes 0 to  $W_\delta$ . If  $v_i$  does not fall on  $A_l$ , then both  $w_l$  and  $w'_l$  must be in the same cloud, so either  $w_l$  and  $w'_l$  contribute 1 to  $W_\delta$  and  $W'_\delta$ , respectively, or each contributes 0 to  $W_\delta$  and  $W'_\delta$ , respectively. Finally, note that each  $w_i$  and  $w'_i$  for  $i > k$  must fall in the same cloud as the basepoint  $*$  in both  $\Phi$  and  $\Psi$  since  $\Gamma$  is sufficiently subdivided, so in this case  $w_i$  and  $w'_i$  contribute 1 to  $W_0$  and  $W'_0$ , respectively.

In other words, the difference  $W_\delta - W'_\delta$  is equal to the number of arcs oriented towards  $v_i$  in the direction  $\delta$  minus the number of arcs oriented away from  $v_i$  in the direction  $\delta$ . This is the number  $\eta_\delta(v_i)$  in Definition 2.3, so by assumption, there must be at least one direction  $\delta$  such that  $W_\delta \neq W'_\delta$ . Furthermore, by the remarks following the same definition, there must be directions  $\delta$  and  $\delta'$  such that  $W'_\delta - W_\delta > 0$  and  $W'_{\delta'} - W_{\delta'} < 0$ , and therefore,  $g(C_\delta) > f(C_\delta)$  and  $g(C_{\delta'}) < f(C_{\delta'})$ . In this case, call  $C_\delta$  and  $C'_{\delta'}$  positive and negative clouds, respectively, to reflect that  $[d_i]$  has more vertices than  $[c_i]$  in direction  $\delta$  (from  $v_i$ ) and less vertices in direction  $\delta'$ . It is possible that either all 3 clouds are categorized as either positive or negative or that one cloud remains uncategorized. But, in either case, we have  $[c_i] \neq [d_i]$ . As an example, the figure below shows the classes  $[c_4]$  and  $[d_4]$  with  $\Gamma$  as above.

Furthermore, it follows from this description that  $[c_i] \neq [d_j]$  for any  $i, j$ . Indeed, if  $[c_i] = [d_j]$ , then both classes must contain a common edge. But, this can only

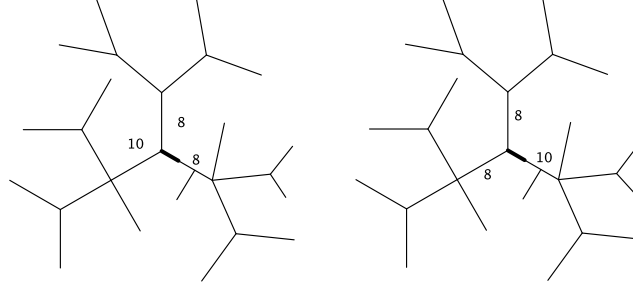


FIGURE 7. Cloud representations for  $[c_4]$  (left) and  $[d_4]$  (right) corresponding to the cells  $\Phi$  and  $\Psi$  in  $UD^{27}(\Gamma)$

happen if  $i = j$  and  $t_i = 3$ , but we just saw  $[c_i] \neq [d_i]$  in this case. So, the cells  $\Phi$  and  $\Psi$  satisfy the hypotheses of Lemma 3.6, proving statement 1.

The construction for statement 2 is similar. Here, by Theorem 2.5,  $UD^n(\Gamma)$  is homotopic to a complex of dimension  $q$ . Let  $v_1, \dots, v_r$  be the vertices of degree greater than 3, and let  $\tilde{v}_1, \dots, \tilde{v}_s$  be the vertices of degree 3. We consider the 2 cases in the statement:

**Case 2a**,  $s \geq 2(q - r)$ : Let  $R = \min\{r, q\}$ , and for  $i = 1, \dots, R$ , let  $e_i$ ,  $u_i$ ,  $e'_i$ , and  $u'_i$  be as in the proof of statement 1 (so that  $e_i, e'_i$  are order-disrespecting edges in directions 2 and 3 from  $v_i$ , and  $u_i, u'_i$  are blocked vertices in directions 1 and 2). If  $r < q$ , for  $i = 1, \dots, 2(q - r)$ , let  $\tilde{e}_i$  be the edge in direction 2 from  $\tilde{v}_i$ , and let  $\tilde{u}_i$  be the vertex on the edge in direction 1 from  $\tilde{v}_i$  which forces  $\tilde{e}_i$  to be order-disrespecting. If  $\epsilon = 0$ , let

$$\Phi = \{e_1, \dots, e_R, u_1, \dots, u_R, \tilde{e}_1, \dots, \tilde{e}_{q-r}, \tilde{u}_1, \dots, \tilde{u}_{q-r}\},$$

$$\Psi = \{e'_1, \dots, e'_R, u'_1, \dots, u'_R, \tilde{e}_{q-r+1}, \dots, \tilde{e}_{2(q-r)}, \tilde{u}_{q-r+1}, \dots, \tilde{u}_{2(q-r)}\},$$

where the edges  $\tilde{e}_i$  and vertices  $\tilde{u}_i$  do not appear if  $r \geq q$ . If  $\epsilon = 1$ , add the vertex  $*$  to each cell. An example is below for  $\Gamma$  as above with  $q = 5$ ,  $\epsilon = 0$ :

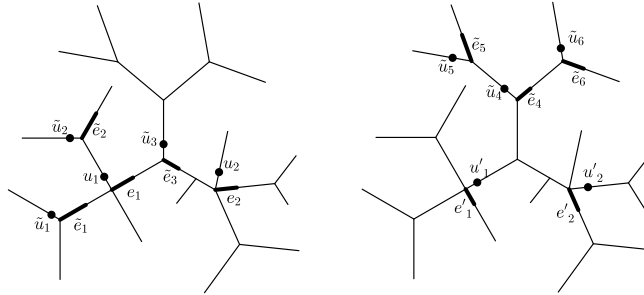


FIGURE 8. The critical cells  $\Phi$  and  $\Psi$  in  $UD^{10}(\Gamma)$

For the same reasons as in the first part of the proof, the cells  $\Phi$  and  $\Psi$  are critical, and if  $\{[c_1], \dots, [c_q]\}$  and  $\{[d_1], \dots, [d_q]\}$  are the collections of equivalence classes of 1-cells having  $[\Phi]$  and  $[\Psi]$  as their least upper bounds, here it is clear that  $[c_i] \neq [d_j]$  for any  $i, j$  since no edge in  $\Phi$  is in  $\Psi$ .

**Case 2b,  $s < 2(q - r)$ :** First, consider the  $\epsilon = 0$  case. Let  $\{A_i\}_{i=1}^k$  and  $\mathcal{V}$  be as in the statement. For  $l = 1, \dots, k$ , let  $V_l$  and  $V'_l$  be the initial and terminal vertices of the arc  $A_l$  (again with respect to the orientations of  $A_l$ ). Let  $E_l$  be the edge in direction 2 from  $V_l$ , and let  $U_l$  be the blocked vertex in direction 1 from  $V'_l$  which forces  $E_l$  to be order-disrespecting. Define  $E'_l$  and  $U'_l$  similarly (so that  $E'_l$  is in direction 2 from  $V'_l$  and  $U'_l$  is in direction 1). Note we must have  $2k \leq m - r'$  so  $s + r = m \geq 2k + r'$ , and also  $r \geq r'$ , so by assumption, we have  $2q > s + 2r = s + r + r \geq 2k + r' + r \geq 2k + 2r'$ , and therefore  $q - r' - k > 0$ .

Let  $\widehat{V}_1, \dots, \widehat{V}_{r'}$  be the vertices of degree greater than 3 which aren't endpoints of any arc  $A_l$ . Define  $\widehat{E}_i$  and  $\widehat{U}_i$  analogously to the definitions of  $E_l$  and  $U_l$ . Let  $\widehat{E}'_i$  be the edge in direction 3 from  $\widehat{V}_i$ , and let  $\widehat{U}'_i$  be the vertex in direction 2 which forces  $\widehat{E}'_i$  to be order-disrespecting. Let  $\widetilde{V}_1, \dots, \widetilde{V}_{q-r'-k}$  be the first  $q - r' - k$  vertices in  $\mathcal{V}$ . Define  $\widetilde{E}_i$  and  $\widetilde{U}_i$  analogously to  $E_i$  and  $U_i$ . Let

$$\Phi = \{E_1, \dots, E_k, U_1, \dots, U_k, \widehat{E}_1, \dots, \widehat{E}_{r'}, \widehat{U}_1, \dots, \widehat{U}_{r'}, \widetilde{E}_1, \dots, \widetilde{E}_{q-r'-k}, \widetilde{U}_1, \dots, \widetilde{U}_{q-r'-k}\},$$

$$\Psi = \{E'_1, \dots, E'_k, U'_1, \dots, U'_k, \widehat{E}'_1, \dots, \widehat{E}'_{r'}, \widehat{U}'_1, \dots, \widehat{U}'_{r'}, \widetilde{E}_1, \dots, \widetilde{E}_{q-r'-k}, \widetilde{U}_1, \dots, \widetilde{U}_{q-r'-k}\}.$$

An example is below for  $q = 8, \epsilon = 0, r' = 2$ , and  $k = 3$ . The set  $\mathcal{V}$  consists of the vertices  $\widetilde{V}_1, \widetilde{V}_2$ , and  $\widetilde{V}_3$ , so that  $|\mathcal{V}| = 3 = q - r' - k$ .

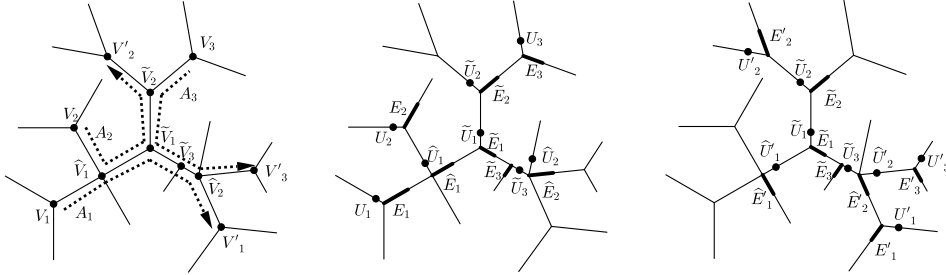


FIGURE 9. The critical cells  $\Phi$  and  $\Psi$  in  $UD^{16}(\Gamma)$  (middle and right)

The cells  $\Phi$  and  $\Psi$  are critical. As above, if  $\{[c_1], \dots, [c_q]\}$  and  $\{[d_1], \dots, [d_q]\}$  are the collections of equivalence classes of 1-cells having  $[\Phi]$  and  $[\Psi]$  as their least upper bounds, and if  $[c_i] = [d_j]$  for some  $i, j$ , then  $[c_i]$  and  $[d_j]$  must have a common edge  $e$  which can only happen if  $e = \widetilde{E}_t$  for some  $t$ , so in particular,  $\widetilde{V}_t$  is of degree 3. For such a  $t$  and each  $\delta = 0, 1, 2$ , let  $C_\delta$  be the cloud in direction  $\delta$  from  $\widetilde{V}_t$ , and let

$$\begin{aligned} M_\delta &= |\{p : V_p \in C_\delta\}|, & M'_\delta &= |\{p : V'_p \in C_\delta\}|, \\ \widehat{M}_\delta &= |\{p : \widehat{V}_p \in C_\delta\}|, & \widetilde{M}_\delta &= |\{p : p \neq t, \widetilde{V}_p \in C_\delta\}|. \end{aligned}$$

Then, if  $f(C_\delta)$  and  $g(C_\delta)$  are the values of  $C_\delta$  in the diagrams for  $\Phi$  and  $\Psi$ , and  $\lambda_\delta$  is as above, we have

$$\begin{aligned} f(C_\delta) &= 2(M_\delta + \widehat{M}_\delta + \widetilde{M}_\delta) + \lambda_\delta, \\ g(C_\delta) &= 2(M'_\delta + \widehat{M}_\delta + \widetilde{M}_\delta) + \lambda_\delta, \end{aligned}$$

so that  $f(C_\delta) - g(C_\delta) = 2(M_\delta - M'_\delta)$ . Arguments similar given to those in the first part of the proof shows that  $M_\delta - M'_\delta = \eta_\delta(\widetilde{V}_t)$ , so again we see that at least two of



the clouds around  $\tilde{V}_t$  can be categorized as either positive or negative, and each of these clouds has a different value in the diagram for  $[c_i]$  than it does in the diagram for  $[d_j]$ , so  $[c_i] \neq [d_j]$ .

For  $\epsilon = 1$ , construct  $\Phi$  and  $\Psi$  as above with the following modifications. First, the vertices  $\tilde{V}_1, \dots, \tilde{V}_{q-r'-k}$  are the first vertices in  $\mathcal{W}' \cup \mathcal{W}$  (where  $k = 0$  and  $r' = r$  if  $s' \geq q - r$ ). Next, since  $n = 2q + 1 \geq 3$ , we can again assume the initial and terminal endpoints of  $A_0$  fall in a cloud in the collection of clouds determined by the edges in  $\Phi$  and  $\Psi$ , respectively. Then, let  $x$  be a vertex in the same cloud as the initial endpoint of  $A_0$  (in the system of clouds determined by the edges of  $\Phi$ ) such that  $x$  is blocked in

$$\Phi = \{E_1, \dots, E_k, U_1, \dots, U_k, \hat{E}_1, \dots, \hat{E}_{r'}, \hat{U}_1, \dots, \hat{U}_{r'}, \tilde{E}_1, \dots, \tilde{E}_{q-r'-k}, \tilde{U}_1, \dots, \tilde{U}_{q-r'-k}, x\},$$

and let  $x'$  be a vertex in the same cloud as the terminal endpoint of  $A_0$  (in the system of clouds determined by the edges of  $\Psi$ ) such that  $x'$  is blocked in

$$\Psi = \{E'_1, \dots, E'_k, U'_1, \dots, U'_k, \hat{E}'_1, \dots, \hat{E}'_{r'}, \hat{U}'_1, \dots, \hat{U}'_{r'}, \tilde{E}'_1, \dots, \tilde{E}'_{q-r'-k}, \tilde{U}'_1, \dots, \tilde{U}'_{q-r'-k}, x'\}.$$

Here, we of course assume that  $x$  is unique from the other vertices in  $\Phi$  and  $x'$  is unique from the other vertices in  $\Psi$ . If  $s' \geq q - r$ , the edges  $E_i$  and  $E'_i$  and the vertices  $U_i$  and  $U'_i$  do not appear in  $\Phi$  and  $\Psi$ .

For example, for  $n = 17$ , we can let the arcs  $A_1$ ,  $A_2$ , and  $A_3$  be as in Figure 9, so  $\mathcal{W} = \{\tilde{V}_1, \tilde{V}_2, \tilde{V}_3\}$ , and trivially let the arc  $A_0$  be the unique edge which has  $*$  as an endpoint, so that  $\mathcal{W}' = \emptyset$ . For a less trivial example, if  $q = 9$ ,  $\epsilon = 1$ , we can let  $A_0$  be the arc  $A_3$  in Figure 9 slightly enlarged so that its interior includes its two original endpoints labeled  $V_3$  and  $V'_3$  in Figure 9, so that here now we have  $k = 2$ ,  $r' = 2$ . See Figure 10. The set  $\mathcal{W}'$  consists of the vertices labeled  $\tilde{V}_1, \dots, \tilde{V}_5$ , and  $\mathcal{W} = \emptyset$  so that  $|\mathcal{W}| = 0 = 9 - 2 - 2 - 5 = q - r' - k - s'$ . Alternatively, we can let  $\mathcal{W}'$  consist of the vertices labeled  $\tilde{V}_3, \tilde{V}_5$  and let  $\mathcal{W}$  consist of the vertices labeled  $\tilde{V}_1, \tilde{V}_2, \tilde{V}_4$  in Figure 10, so that  $s' = 2$  and  $|\mathcal{W}| = 3 = 9 - 2 - 2 - 2 = q - r' - k - s'$ .

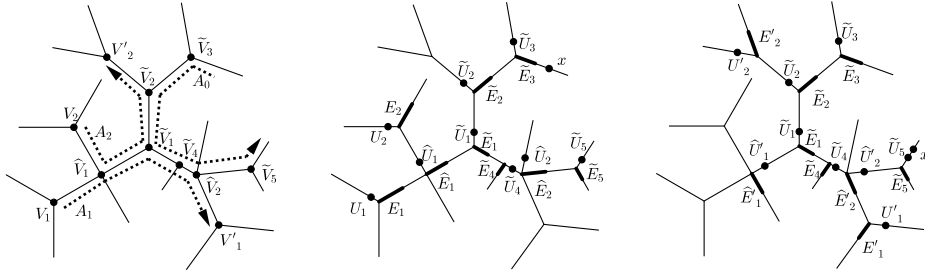


FIGURE 10. The critical cells  $\Phi$  and  $\Psi$  in  $UD^{19}(\Gamma)$  (middle and right)

Again, the cells  $\Phi$  and  $\Psi$  are critical. If  $\{[c_1], \dots, [c_q]\}$  and  $\{[d_1], \dots, [d_q]\}$  are the collections of equivalence classes of 1-cells having  $[\Phi]$  and  $[\Psi]$  as their least upper bounds, and  $[c_i] = [d_j]$  for some  $i, j$ , then again it must be the case that  $[c_i]$  and  $[d_j]$  have a common edge  $\tilde{E}_t$ , so  $\tilde{V}_t$  is of degree 3. Now, with the notation from the  $\epsilon = 0$  case, we have

$$\begin{aligned} f(C_\delta) &= 2(M_\delta + \widehat{M}_\delta + \widetilde{M}_\delta) + \lambda_\delta + \mu_\delta, \\ g(C_\delta) &= 2(M'_\delta + \widehat{M}_\delta + \widetilde{M}_\delta) + \lambda_\delta + \mu'_\delta, \end{aligned}$$

where  $\mu_\delta = 1$  if  $x$  falls in  $C_\delta$ , and  $\mu_\delta = 0$  if  $x$  does not fall in  $C_\delta$ , and similarly,  $\mu'_\delta = 1$  if  $x'$  falls in  $C_\delta$ , and  $\mu'_\delta = 0$  if  $x'$  does not fall in  $C_\delta$ . Since the endpoints of  $A_0$  fall in clouds determined by the edges in  $\Phi$  and  $\Psi$ , the vertex  $\tilde{V}_t$  cannot be an endpoint, so it either does not fall on  $A_0$  or it falls on the interior of  $A_0$ .

If  $\tilde{V}_t$  does not fall on  $A_0$ , so that we have  $\tilde{V}_t \in \mathcal{W}$ , then  $x$  and  $x'$  are in the same cloud in the diagrams for  $[c_i]$  and  $[d_j]$ , so that  $\mu_\delta = \mu'_\delta$  for each  $\delta$ , and therefore  $f(C_\delta) - g(C_\delta) = 2(M_\delta - M'_\delta) = 2\eta_\delta(\tilde{V}_t)$ , and since the collection  $\{A_i\}_{i=1}^k$  is allowable for  $\mathcal{W}$ , we can categorize at least one cloud as positive and one as negative as above, so that  $[c_i] \neq [d_j]$ .

If  $\tilde{V}_t$  falls on the interior of  $A_0$ , then it is possible that  $\tilde{V}_t \in \mathcal{W}$  or  $\tilde{V}_t \in \mathcal{W}'$ , but in either case,  $x$  and  $x'$  must fall in different clouds  $C_{\delta_1}$  and  $C_{\delta_2}$ . Then, we have

$$f(C_{\delta_1}) = 2(M_{\delta_1} + \widehat{M}_{\delta_1} + \widetilde{M}_{\delta_1}) + \lambda_{\delta_1} + 1$$

$$g(C_{\delta_1}) = 2(M'_{\delta_1} + \widehat{M}_{\delta_1} + \widetilde{M}_{\delta_1}) + \lambda_{\delta_1} + 0,$$

so now  $f(C_{\delta_1}) - g(C_{\delta_1}) = 2(M_{\delta_1} - M'_{\delta_1}) + 1$ , which is odd and therefore non-zero. Likewise,  $f(C_{\delta_2}) - g(C_{\delta_2}) = 2(M_{\delta_2} - M'_{\delta_2}) - 1$ , which is again odd and therefore non-zero. Furthermore, since the sum of the values of the clouds around  $\tilde{V}_t$  must equal  $n - 1$  in each cell, we have

$$f(C_0) - g(C_0) + f(C_1) - g(C_1) + f(C_2) - g(C_2) = 0,$$

so that  $g(C_\delta) - f(C_\delta) > 0$  and  $g(C_{\delta'}) - f(C_{\delta'}) < 0$  for two directions  $\delta \neq \delta'$ , so again at least one cloud is categorized as positive and one as negative, so  $[c_i] \neq [d_j]$ .

Therefore, in all cases, the conditions of Lemma 3.6 are met, so

$$TC(UD^n(\Gamma)), TC(D^n(\Gamma)) \geq 2q + 1.$$

This, combined with the upper bounds stated at the beginning of the proof, gives the result.  $\square$

It is worth noting that if we insist no vertices have degree 3, the statement of Theorem 1.2 becomes much simpler and determines the topological complexity for all  $n$  for both configuration spaces, provided they are connected.

**Corollary 3.7.** *Let  $\Gamma$  be a tree with no vertices of degree 3. Let  $k = \min \left\{ \left\lfloor \frac{n}{2} \right\rfloor, m(\Gamma) \right\}$ . Then,  $TC(UC^n(\Gamma)) = 2k + 1$ . Also, if  $n = 1$  or  $m(\Gamma) \geq 1$ , then  $TC(C^n(\Gamma)) = 2k + 1$ .*

*Proof.* Let  $r$  and  $s$  be as in Theorem 1.2, so that  $r = m(\Gamma) =: m$ , and  $s = 0$ . If  $m \geq 1$  and  $n \geq 2m$ , the claim follows from statement 1 of the theorem. If  $n = 2q + \epsilon < 2m = 2r$ , with  $\epsilon \in \{0, 1\}$  and  $q \geq 1$ , then  $r > q$ , so  $2(q - r) < 0 = s$ , and the claim follows from statement 2a. If  $m = 0$ , so that  $\Gamma$  is homeomorphic to a closed interval, then Lemma 3.4 gives  $TC(UD^n(\Gamma)) \leq 1$ , but  $TC(X) \geq 1$  for any space, so,  $TC(UC^n(\Gamma)) = 1$ . For remaining case,  $n = 1$ ,  $m \geq 1$ , we have  $C^n(\Gamma) = UC^n(\Gamma) = \Gamma$ , so all three spaces have topological complexity 1 since  $\Gamma$  is contractible.  $\square$

Now, we discuss how in some sense, the results in Theorem 1.2 are the best we can achieve with the methods used here. Consider the case  $2m \leq n < 2m + k$ , with  $m$  and  $k$  as in the first part of the theorem. In this case, a construction similar to the one given in the proof shows that there is a critical  $m$ -cell, which corresponds to a non-zero  $m$ -dimensional cohomology class, so that the space  $UD^n(\Gamma)$  cannot be homotopic to a space of dimension less than  $m$ , so the dimensional bound given

in Theorem 3.2 cannot improve the bound given by the explicit motion planning algorithm in Lemma 3.4. Likewise, if  $q, r$ , and  $s$  are as in the second part of the theorem, but the appropriate collection of arcs does not exist, then there will still be a critical  $q$ -cell, so again the dimension cannot improve the upper bound. The following shows that Lemma 3.6 cannot be used to get improved lower bounds.

**Proposition 3.8.** *Let  $\Gamma$  be a tree with  $m := m(\Gamma) \geq 1$ .*

- (1) *Let  $k$  be as in statement 1 of Theorem 1.2, and assume there is at least one vertex of degree 3 so that  $k \geq 1$ . Let  $n$  satisfy  $n \geq 2m$ , and consider any two critical  $m$ -cells  $\Phi, \Psi \in UD^n(\Gamma)$ . If  $\{[c_1], \dots, [c_m]\}$  and  $\{[d_1], \dots, [d_m]\}$  are the unique collections of equivalence classes of 1-cells having  $[\Phi]$  and  $[\Psi]$  as their least upper bounds, and for all  $i, j$ , we have  $[c_i] \neq [d_j]$ , then  $n \geq 2m + k$ .*
- (2) *Let  $q, r, s, \epsilon$  be as in statement 2b of Theorem 1.2, and consider critical  $q$ -cells  $\Phi, \Psi \in UD^n(\Gamma)$ . If  $\{[c_1], \dots, [c_q]\}$  and  $\{[d_1], \dots, [d_q]\}$  are the unique collections of equivalence classes of 1-cells having  $[\Phi]$  and  $[\Psi]$  as their least upper bounds and for all  $i, j$ , we have  $[c_i] \neq [d_j]$ , then*
  - (a) *if  $\epsilon = 0$ , there is some  $k \geq 1$  such that there exists oriented arcs  $A_1, \dots, A_k$  with the following properties:*
    - (i) *The endpoints of each  $A_l$  are (distinct) essential vertices, neither of which is an endpoint of any other  $A_{l'}$ ,*
    - (ii) *There are  $r' \leq r$  vertices of degree greater than 3 which are not the endpoints of any  $A_l$ ,*
    - (iii) *There is a collection  $\mathcal{V}$  of degree-3 vertices, with  $|\mathcal{V}| \geq q - r' - k$  such that  $\{A_i\}_{i=1}^k$  is allowable for  $\mathcal{V}$ .*
  - (b) *if  $\epsilon = 1$ , there is an arc  $A_0$  whose endpoints have no restrictions and whose interior includes a collection  $\mathcal{W}'$  of  $s' \leq q$  distinct vertices of degree 3, and if  $s' < q - r$ , there are arcs  $A_1, \dots, A_k$ , as above whose endpoints are also not vertices in  $\mathcal{W}'$ , and there is another collection of degree-3 vertices,  $\mathcal{W}$ , such that  $\mathcal{W} \cap \mathcal{W}' = \emptyset$ ,  $|\mathcal{W}| \geq q - r' - k - s'$  and  $\{A_i\}_{i=1}^k$  is allowable for  $\mathcal{W}$ , where  $r'$  is as above.*

*Proof.* We use the contrapositive for both statements 1 and 2. For the first statement, let  $\Phi$  and  $\Psi$  be critical  $m$ -cells, and assume  $2m \leq n < 2m + k$ . For each essential vertex  $v$ , each cell must contain exactly one edge  $e$  having  $\iota(e) = v$  and a blocked vertex  $u$  which makes  $e$  order-disrespecting. There is no choice for  $e$  or  $u$  if  $v$  is of degree 3.

If  $n > 2m$ , let  $w_1, \dots, w_{n-2m}$  be the remaining vertices in  $\Phi$ , and let  $w'_1, \dots, w'_{n-2m}$  be the remaining vertices in  $\Psi$ . For  $l = 1, \dots, n - 2m < k$ , let  $A_l$  be the geodesic from  $w_l$  to  $w'_l$ , oriented so that  $w_l$  is the initial endpoint and  $w'_l$  is the terminal endpoint if  $w_l \neq w'_l$ . If  $w_l = w'_l$ , extend  $A_l$  slightly so that it is a small arc starting at  $w_l$  which doesn't intersect any essential vertices. This gives a collection of less than  $k$  oriented arcs in  $\Gamma$ , so by the minimality of  $k$ , this collection cannot be allowable for the set of vertices of degree 3 in  $\Gamma$ . It is not possible that a degree-3 vertex  $v$  is an endpoint of any  $A_l$ , since no  $w_l$  or  $w'_l$  can be essential. Therefore, there is some degree-3 vertex  $v$  which has the property  $\eta_0(v) = \eta_1(v) = \eta_2(v) = 0$ . Let  $e$  be the edge in direction 2 from  $v$ . This edge must be in both  $\Phi$  and  $\Psi$ . Suppose  $i$  and  $j$  have the property that  $e$  is the unique edge in  $[c_i]$  and  $[d_j]$ . Then, in the notation from the proof of Theorem 1.2, for  $\delta = 0, 1, 2$  and each cloud  $C_\delta$  in the diagrams for  $[c_i]$  and  $[d_j]$ , we have  $f(C_\delta) = 2m_\delta + W_\delta + \lambda_\delta$  and  $g(C_\delta) = 2m_\delta + W'_\delta + \lambda_\delta$ . But,

as above, we have  $W_\delta - W'_\delta = \eta_\delta(v) = 0$  for each  $\delta$ , and therefore  $W_\delta = W'_\delta$  and  $f(C_\delta) = g(C_\delta)$ , for each  $\delta$ , so  $[c_i] = [d_j]$ .

If  $n = 2m$ , then for each vertex  $v$  of degree 3, the edge  $e$  in direction 2 from  $v$  must appear in both cells, so again if  $[c_i]$  and  $[d_j]$  are the equivalence classes which contain the edge  $e$ , we now have  $f(C_\delta) = 2m_\delta + \lambda_\delta = g(C_\delta)$  for each  $\delta = 0, 1, 2$ , so  $[c_i] = [d_j]$ .

For statement 2, assume that the collections of arcs and vertices in 2a or 2b do not exist for the appropriate value of  $\epsilon$ . Note first that if  $\epsilon = 0$ , any critical  $q$ -cell must consist solely of edges  $e_1, \dots, e_q$  and blocked vertices  $u_1, \dots, u_q$  which force the edges  $e_1, \dots, e_q$  to be order-disrespecting. In particular, each  $e_i$  has the property that  $\iota(e_i)$  is an essential vertex. If  $\epsilon = 1$ , then the same is true, except that the cell contains one additional blocked vertex. Let  $\Phi$  and  $\Psi$  be critical  $q$ -cells, and let  $\mathcal{S}$  (resp.  $\mathcal{S}'$ ) be the set of essential vertices  $v$  such that  $\iota(e) = v$  for some  $e$  in  $\Phi$  (resp.  $\Psi$ ), so  $|\mathcal{S}| = |\mathcal{S}'| = q$ .

Let  $\mathcal{T} = \mathcal{S} \setminus (\mathcal{S} \cap \mathcal{S}')$  and  $\mathcal{T}' = \mathcal{S}' \setminus (\mathcal{S} \cap \mathcal{S}')$ , so  $|\mathcal{T}| = |\mathcal{T}'| =: k$ . The fact that  $s < 2(q - r)$  implies that at least one vertex of degree 3 appears in  $\mathcal{S} \cap \mathcal{S}'$ . Let  $\tilde{V}_1, \dots, \tilde{V}_{\bar{s}}$  be the vertices of degree 3 in  $\mathcal{S} \cap \mathcal{S}'$ , so that  $\bar{s} > 0$  and let  $\hat{V}_1, \dots, \hat{V}_{\bar{r}}$  be the vertices of degree greater than 3 in  $\mathcal{S} \cap \mathcal{S}'$ . The edge in direction 2 from each vertex  $\tilde{V}_t$  must be included in both  $\Phi$  and  $\Psi$ . In what follows,  $[c_i]$  and  $[d_j]$  are the equivalence classes which contain the edge in direction 2 from whichever vertex  $\tilde{V}_t$  is being discussed.

If  $\epsilon = 1$ , let  $x$  (resp.  $x'$ ) be the additional vertex in  $\Phi$  (resp.  $\Psi$ ). Let  $A_0$  be the geodesic from  $x$  to  $x'$  and extend  $A_0$  slightly if  $x \neq x'$  so that it is a small arc which doesn't intersect any vertex in  $\mathcal{S}$  or  $\mathcal{S}'$ . Let  $\mathcal{W}'$  be the set of degree-3 vertices in  $\mathcal{S} \cap \mathcal{S}'$  which fall on the interior of  $A_0$ , and let  $s' = |\mathcal{W}'|$ . By the assumption of the non-existence of the appropriate collections of arcs and vertices, we have  $s' < q - r$ .

First consider the case  $k = 0$ . Here, we have  $\mathcal{S} = \mathcal{S}'$ , and  $\bar{s} + \bar{r} = q$ . If  $\epsilon = 0$ , then for each vertex  $\tilde{V}_t \in \mathcal{S} \cap \mathcal{S}' = \mathcal{S} = \mathcal{S}'$ , we have, with the notation from the proof of Theorem 1.2,  $f(C_\delta) = 2(\tilde{M}_\delta + \hat{M}_\delta) + \lambda_\delta = g(C_\delta)$ , so  $[c_i] = [d_j]$ . If  $\epsilon = 1$ , since we have  $s' < q - r$ , there must be at least one vertex  $\tilde{V}_t$  which does not fall on  $A_0$ . Then,  $x$  and  $x'$  must fall in the same cloud in the system of clouds determined by the edge in direction 2 from  $\tilde{V}_t$ . Again using the notation from the proof of Theorem 1.2, we have  $\mu_\delta = \mu'_\delta$ , so  $f(C_\delta) = 2(\tilde{M}_\delta + \hat{M}_\delta) + \lambda_\delta + \mu_\delta = 2(\tilde{M}_\delta + \hat{M}_\delta) + \lambda_\delta + \mu'_\delta = g(C_\delta)$ , and therefore  $[c_i] = [d_j]$ .

If  $k \geq 1$ , let  $V_1, \dots, V_k$  and  $V'_1, \dots, V'_k$  be the vertices in  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively, and let  $A_l$  be the oriented geodesic from  $V_l$  to  $V'_l$ . This gives a collection of  $k$  arcs with  $2k$  distinct essential endpoints if  $\epsilon = 0$ , and one additional arc  $A_0$  if  $\epsilon = 1$ . Also note that  $\bar{s} + \bar{r} + k = q$  and  $\bar{r} \leq r'$ , where  $r'$  is the number of vertices of degree greater than 3 which are not endpoints of any  $A_l$ , so  $\bar{s} = q - \bar{r} - k \geq q - r' - k$ .

For  $\epsilon = 0$ , let  $\mathcal{V} = \{\tilde{V}_1, \dots, \tilde{V}_{\bar{s}}\}$ , so that  $|\mathcal{V}| = \bar{s} \geq q - r' - k$ . By assumption, the arcs  $\{A_i\}_{i=1}^k$  cannot be allowable for  $\mathcal{V}$ . Any vertex  $v \in \mathcal{V}$  cannot be the endpoint of any  $A_l$  since no endpoint is in  $\mathcal{S} \cap \mathcal{S}'$ . So, there is at least one degree-3 vertex  $\tilde{V}_t$  such that  $\eta_0(\tilde{V}_t) = \eta_1(\tilde{V}_t) = \eta_2(\tilde{V}_t) = 0$ . So, we have  $f(C_\delta) - g(C_\delta) = 2(M_\delta - M'_\delta) = 2\eta_\delta(v) = 0$ , for each  $\delta \in \{0, 1, 2\}$ , so each cloud has the same value in the diagram for  $[c_i]$  as it does in the diagram for  $[d_j]$ . So,  $[c_i] = [d_j]$ .

For  $\epsilon = 1$ , let  $\mathcal{W}$  be the set of degree-3 vertices in  $(\mathcal{S} \cap \mathcal{S}') \setminus \mathcal{W}'$ , so that  $\mathcal{W} \cap \mathcal{W}' = \emptyset$  and  $|\mathcal{W}| = \bar{s} - s' \geq q - r' - k - s'$ , and therefore, again by assumption,  $\{A_i\}_{i=1}^k$

is not allowable for  $\mathcal{W}$ . No vertex in  $\mathcal{W}$  can be an endpoint of any arc  $A_t$ , so there must be some vertex  $\tilde{V}_t \in \mathcal{W}$  with  $\eta_\delta(\tilde{V}_t) = 0$  for  $\delta = 0, 1, 2$ . Now, we have  $f(C_\delta) - g(C_\delta) = 2(M_\delta - M'_\delta) + \mu_\delta - \mu'_\delta$ , but since  $\tilde{V}_t$  is not on  $A_0$ , we must have both  $x$  and  $x'$  in the same cloud in the system of clouds determined by the edge in direction 2 from  $\tilde{V}_t$ , so  $\mu_\delta = \mu'_\delta$ , and therefore  $f(C_\delta) - g(C_\delta) = 2(M_\delta - M'_\delta) = 2\eta_\delta(\tilde{V}_t) = 0$  for each direction  $\delta$ , so again we have  $[c_i] = [d_j]$ .  $\square$

To give a better idea of the values of  $n$  for which Theorem 1.2 determines  $TC(UC^n(\Gamma))$  (and  $TC(C^n(\Gamma))$ ), we consider all values of  $n \geq 2$  with  $\Gamma$  as in the left of Figure 6, where we have  $m = 11$ ,  $r = 2$ , and  $s = 9$ . If  $n$  is either sufficiently large or sufficiently small, it is easy to determine if the theorem applies. For  $n \geq 25$ , statement 1 of the theorem applies, and for  $2 \leq n \leq 13$ , statement 2a applies. For  $22 \leq n \leq 24$ , the theorem does not apply. For the “middle” values of  $n$ , namely  $14 \leq n \leq 21$ , only statement 2b might apply.

For  $n = 14$  statement 2b does apply, since we may chose the arcs  $A_1$ ,  $A_2$ , and  $A_3$  as in Figure 9, and let  $\mathcal{V}$  consist of the vertices  $\tilde{V}_1$ ,  $\tilde{V}_2$ , and  $\tilde{V}_3$  (as labeled in Figure 9). Then, we have  $k = 3$ ,  $r' = 2$ ,  $|\mathcal{V}| = 3 \geq q - r' - k = 7 - 2 - 3 = 2$ , and the arcs  $A_1$ ,  $A_2$ , and  $A_3$  are again allowable for  $\mathcal{V}$ . Note the vertex  $\tilde{V}_3$  would not be used in the construction of the cells  $\Phi$  and  $\Psi$  for  $n = 14$ . For  $n = 15$ , we can again let  $A_0$  be the unique edge which has  $*$  as one of its endpoints, and keep  $A_1$ ,  $A_2$ , and  $A_3$  the same, so the theorem also applies for  $n = 15$ . The cases  $n = 16$ ,  $n = 17$ , and  $n = 19$  are covered in the proof.

For  $n = 18$ , suppose the appropriate collection of arcs  $A_1, \dots, A_k$  exists and is allowable for a set  $\mathcal{V}$  of vertices of degree 3, so that we have  $|\mathcal{V}| \geq q - r' - k$ . On the other hand, there must be a total of  $2k$  distinct vertices which are the endpoints of the arcs. No vertex in  $\mathcal{V}$  can be any of these endpoints, so we must have  $|\mathcal{V}| \leq s - 2k + r - r'$ , since there are  $r - r'$  vertices of degree greater than 3 which are the endpoint of some arc, and the remaining  $2k - (r - r')$  endpoints must be of degree 3. So, we have

$$9 - r' - k = q - r' - k \leq |\mathcal{V}| \leq s - 2k + r - r' = 9 - 2k + 2 - r'.$$

Comparing the left and right sides, we see that we must have  $k \leq 2$ . Since we also must have  $r' \leq r = 2$ , this gives  $|\mathcal{V}| \geq 9 - 2 - 2 = 5$ . But, the only vertices of degree 3 which can fall on the interior of any arc with essential endpoints are the vertices labeled  $v_4$ ,  $v_5$ , and  $v_8$  in Figure 6. so  $|\mathcal{V}| \leq 3$ , arriving at a contradiction, so the theorem does not apply for  $n = 18$ , and for similar reasons, the theorem does not apply for  $n = 20$ .

For  $n = 21$ , suppose the appropriate collection of arcs  $A_0, A_1, \dots, A_k$  and collections of vertices  $\mathcal{W}$  and  $\mathcal{W}'$  exists, so that  $|\mathcal{W} \cup \mathcal{W}'| \geq q - r' - k = 10 - r' - k$ . Similar to above, since no vertex in  $\mathcal{W} \cup \mathcal{W}'$  can be an endpoint of any arc  $A_1, \dots, A_k$ , we must have

$$10 - r' - k = q - r' - k \leq |\mathcal{W} \cup \mathcal{W}'| \leq s - 2k + r - r' = 9 - 2k + 2 - r',$$

so that  $k \leq 1$ . Since again we must have  $r' \leq 2$ , this gives  $|\mathcal{W} \cup \mathcal{W}'| \geq 10 - r' - k \geq 10 - 2 - 1 = 7$ . However, again the only vertices of degree 3 which can fall on the interior of any arc  $A_1$  are the vertices labeled  $v_4$ ,  $v_5$ , and  $v_8$  in Figure 6, and here it is clear that at most two additional vertices could be included in the interior of the arc  $A_0$ , so that  $|\mathcal{W} \cup \mathcal{W}'| \leq 5$ , a contradiction, so the theorem does not apply for  $n = 21$ .

So, the only values of  $n \geq 2$  for which Theorem 1.2 does not determine  $TC(UC^n(\Gamma))$  and  $TC(C^n(\Gamma))$  are  $n = 18$  and  $20 \leq n \leq 24$ .

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